

ADDING MACHINE MAPS AND MINIMAL SETS FOR ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, we focus attention on extending the topological conjugacy of adding machine maps and minimal systems to iterated function systems. We provide necessary and sufficient conditions for an iterated function system to be conjugated to an adding machine map. It is proved that every minimal iterated function system which has some non-periodic regular point is semi-conjugate to an adding machine map. Furthermore, we investigate the topological conjugacy of an infinite family of tent maps, as well as the restriction of a map to its ω -limit set with an iterated function system.

1. INTRODUCTION

Minimality is an important concept in the study of dynamical systems. It describes the irreducibility of a system from the topological point of view, which means that a minimal system has no proper subsystem [4]. Adding machines are a fundamental component of discrete dynamical systems [1, 5, 6, 7]. Let $f : X \rightarrow X$ be a continuous map of a compact metric space X . In [5] the authors give sufficient and necessary conditions for f to be topologically conjugate to an adding machine map.

Let us recall that an Iterated Function System(**IFS**) $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is

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any family of continuous mappings $f_\lambda : X \rightarrow X$, $\lambda \in \Lambda$, where Λ is a finite nonempty set (see[2]).

Iterated function systems are used for the construction of deterministic fractals and have found numerous applications, in particular to image compression and image processing [2]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing have been extended to **IFS** (see [3, 11, 12, 8]). The present paper concerns the minimality and adding machine maps for **IFS**. In the next section, we introduce certain notation and we give some definitions, which will be used throughout the paper. In Section 3 we proceed with the study of minimality and minimal sets for **IFS**. Theorem 3.5, which is the main result of this section, shows that for a sequence $\alpha = (j_1, j_2, \dots)$ of integers greater than 1, an **IFS** is topologically conjugate to adding machine map g_α , if and only if conditions (1) – (3) of Theorem 3.5 are satisfied. In Section 4 several lemmas lead to Theorem 4.6 which shows that: If the **IFS**, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, X is infinite and \mathcal{F} has some non-periodic regularly recurrent points, then there is a sequence α of prime numbers and a continuous surjective map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f_\lambda$, for all $\lambda \in \Lambda$. Also, we will present several corollaries of this theorem and its proof.

2. PRELIMINARIES

Let (X, d) be a compact metric space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an **IFS**. We let $\Lambda^{\mathbb{Z}^+}$ denote the set of all infinite sequences $\{\lambda_i\}_{i \geq 1}$ of symbols belonging to Λ . For a given $\sigma = \{\lambda_n\} \in \Lambda^{\mathbb{Z}^+}$ we put $\mathcal{F}_\sigma = \{\mathcal{F}_{\sigma_n}\}$ where $\mathcal{F}_{\sigma_n} = f_{\lambda_n} \circ \dots \circ f_{\lambda_2} \circ f_{\lambda_1}$, $n \geq 1$. A sequence $\{x_n\}_{n \geq 1}$ in X is called an orbit of the **IFS** \mathcal{F} , if there exists $\sigma \in \Lambda^{\mathbb{Z}^+}$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for each $\lambda_n \in \sigma$.

For a subset $A \subset X$, we let $\mathcal{F}(A) = \bigcup_{\lambda \in \Lambda} f_\lambda(A)$.

Definition 2.1. Suppose that (X, d) and (Y, d') are compact metric spaces. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two **IFS** such that $f_\lambda : X \rightarrow X$ and $g_\lambda : Y \rightarrow Y$ are continuous maps, for all $\lambda \in \Lambda$. Consider $\sigma \in \Lambda^{\mathbb{Z}_+}$, we say that \mathcal{F}_σ is topologically conjugate to \mathcal{G}_σ if there is a homeomorphism $h : X \rightarrow Y$ such that $\mathcal{G}_{\sigma_n} \circ h = h \circ \mathcal{F}_{\sigma_n}$, for all $n \geq 1$.

By the above definition, for $\sigma \in \Lambda^{\mathbb{Z}_+}$ we say that \mathcal{F}_σ is topologically conjugate to the map $g : Y \rightarrow Y$, if there is a homeomorphism $h : X \rightarrow Y$ such that $g^n \circ h = h \circ \mathcal{F}_{\sigma_n}$, for all $n \geq 1$.

Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an **IFS** and k be a positive integer. Set $\mathcal{F}^k = \{X; f_{\lambda_k} \circ \dots \circ f_{\lambda_1} | \lambda_1, \dots, \lambda_k \in \Lambda\}$.

It should be noted that for every $k > 1$, \mathcal{F}^k is an **IFS** such that its functions and their number are different from those in \mathcal{F} .

Definition 2.2. [9] Let n be a positive integer and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$. A nonempty closed set $M \subseteq X$ is called a \mathcal{F}^n -minimal set if $\mathcal{F}_{\sigma_n}(M) = M$ and $\mathcal{F}_{\sigma_n}(A) \neq A$, for all nonempty sets $A \subset M$ and all $\sigma \in \Lambda^{\mathbb{Z}_+}$. The **IFS** \mathcal{F} is minimal if the only minimal set for \mathcal{F}^1 is the whole space X .

Equivalently, for a minimal iterated function system $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$, for any $x \in X$ the collection of iterates $f_{\lambda_k} \circ \dots \circ f_{\lambda_1}(x)$, $k > 0$ and $\lambda_1, \dots, \lambda_k \in \Lambda$, is dense in X .

Definition 2.3. (Adding machine map. [5]) Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers where each $j_i \geq 2$. Let Δ_α denote all sequences (r_1, r_2, \dots) where $r_i \in \{0, 1, \dots, j_i - 1\}$ for each i . We put a metric d_α on Δ_α given by $d_\alpha((r_1, r_2, \dots), (s_1, s_2, \dots)) = \sum_{i=1}^{\infty} \frac{\delta(r_i, s_i)}{2^i}$, where $\delta(r_i, s_i) = 1$ if $r_i \neq s_i$ and $\delta(r_i, s_i) = 0$ if $r_i = s_i$. Addition in Δ_α is defined as follows:

$$(r_1, r_2, \dots) + (s_1, s_2, \dots) = (z_1, z_2, \dots),$$

where $z_1 = (r_1 + s_1) \bmod j_1$ and $z_1 = (r_2 + s_2 + t_1) \bmod j_2$. Here $t_1 = 0$ if $r_1 + s_1 < j_1$ and $t_1 = 1$ if $r_1 + s_1 \geq j_1$. Continue adding and carrying in this way for the whole sequence.

We define, the adding machine map, $g_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ by

$$g_\alpha((r_1, r_2, \dots)) = (r_1, r_2, \dots) + (1, 0, 0, \dots).$$

If each $j_i = 2$ then the system is called the dyadic adding machine.

Let $f : X \rightarrow X$ be a continuous map. A point x is non-wandering point of f if for any neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points of f is denoted as $\Omega(f)$, if $\Omega(f) = X$ then the systems is said to be non-wandering.

A sequence $\{x_n\}_{n \geq 0}$ is called a δ -pseudo-orbit ($\delta \geq 0$) of f if $d(f(x_n), x_{n+1}) \leq \delta$ for all $n \geq 0$. A map f is said to have the shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that for every δ -pseudo-orbit $\{x_n\}_{n \geq 0}$ of f , there exists $y \in X$ satisfying $d(f^n(y), x_n) < \epsilon$ for all $n \geq 0$.

A dynamical system (X, f) is sensitive if there is $\delta > 0$ with the property that for every nonempty open set $U \subset X$, there is $n > 0$ such that $\text{diam}(f^n(U)) > \delta$ [15].

3. TOPOLOGICAL CONJUGACY OF ADDING MACHINE MAPS AND MINIMAL IFS

In this section we present some lemmas and notations needed in the proof of our main results.

Lemma 3.1. *Let $n \geq 1$ and M be a \mathcal{F}^n -minimal set, then*

$\mathcal{F}_{\sigma_n}(\mathcal{F}^i(M)) = \mathcal{F}^i(M)$, for every positive integer i and every $\sigma \in \Lambda^{\mathbb{Z}_+}$. Also, there is $t \leq n$ such that $\mathcal{F}^i(M)$ is an \mathcal{F}^n -minimal set, for all $i \leq t - 1$.

Proof. By the definition of \mathcal{F}^n -minimal set, it is enough to prove the case of $1 \leq i \leq n-1$. Fix $1 \leq i \leq n-1$. Suppose that $\{\lambda_1, \dots, \lambda_i\}$ and $\{\lambda_{i+1}, \dots, \lambda_{n+i}\}$ are arbitrary sequences in Λ of length i and n , respectively. Since M is a \mathcal{F}^n -minimal set, we have

$$\begin{aligned} f_{\lambda_{n+i}} \circ \dots \circ f_{\lambda_{i+1}}(f_{\lambda_i} \circ \dots \circ f_{\lambda_1}(M)) &= f_{\lambda_{n+i}} \circ \dots \circ f_{\lambda_{n+1}}(\mathcal{F}_{\sigma_n}(M)) \\ &= f_{\lambda_{n+i}} \circ \dots \circ f_{\lambda_{n+1}}(M) \\ &\subseteq \mathcal{F}^i(M). \end{aligned}$$

This implies that, $\mathcal{F}_{\sigma_n}(\mathcal{F}^i(M)) = \mathcal{F}^i(M)$, for all $\sigma \in \Lambda^{\mathbb{Z}_+}$.

Consider $t \leq n$ as the least positive integer with $\mathcal{F}^t(M) \cap M \neq \emptyset$. By contradiction, suppose that $\mathcal{F}_{\sigma_n}(A) = A$, for a nonempty set $A \subset \mathcal{F}^i(M)$ and a sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$. Consider $x \in M$ and the sequence $\{\lambda_1, \dots, \lambda_i\}$ such that $f_{\lambda_i} \circ \dots \circ f_{\lambda_1}(x) \in A$. Then $\mathcal{F}_{\sigma_n}(\mathcal{F}^i(M)) = \mathcal{F}^i(M)$ implies that $f_{\lambda_{n+i}} \circ \dots \circ f_{\lambda_{i+1}}(\mathcal{F}_{\sigma_i}(x)) = f_{\lambda_{n+i}} \circ \dots \circ f_{\lambda_{n+1}}(\mathcal{F}_{\sigma_n}(x)) \in A$. Since $\mathcal{F}_{\sigma_n}(M) = M$, we have $f_{\lambda_n} \circ \dots \circ f_{\lambda_1}(x) \in M$ consequently, $\mathcal{F}^i(M) \cap M \neq \emptyset$. This is a clear contradiction of the fact that $i \leq t-1$. \square

Lemma 3.2. *Let X be a compact Hausdorff space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be a minimal IFS with continuous functions $f_\lambda : X \rightarrow X$. Let n be a positive integer. Then for some \mathcal{F}^n -minimal set M and some $t \leq n$, we have the following properties:*

- (1) X is disjoint union of $M, \mathcal{F}(M), \dots, \mathcal{F}^{t-1}(M)$.
- (2) Each of the sets $M, \mathcal{F}(M), \dots, \mathcal{F}^{t-1}(M)$ is clopen.
- (3) The family $\{M, \mathcal{F}(M), \dots, \mathcal{F}^{t-1}(M)\}$ is the collection of all subsets of X that are \mathcal{F}^t -minimal and also the collection of all subsets of X that are \mathcal{F}^n -minimal.

Proof. Let M be a \mathcal{F}^n -minimal set. By Lemma 3.1, for every positive integer i , $\mathcal{F}^i(M)$ is a \mathcal{F}^n -minimal set. Consider $t \leq n$ as the least positive integer

with $\mathcal{F}^t(M) \cap M \neq \emptyset$. Since minimal sets are disjoint or equal, we have $\mathcal{F}^t(M) = M$. By the choice of t , the sets $M, \mathcal{F}(M), \dots, \mathcal{F}^{t-1}(M)$ are pairwise disjoint. Let $U = M \cup \mathcal{F}(M) \cup \dots \cup \mathcal{F}^{t-1}(M)$, Our choice of t and M shows that, $f_\lambda(U) \subset U$ for all $\lambda \in \Lambda$. So $U = X$ and the statement (1) holds .

Statements (2) and (3) hold trivially whenever X is the disjoint union of the sets $M, \mathcal{F}(M), \dots, \mathcal{F}^{t-1}(M)$. \square

Corollary 3.3. *Let \mathcal{F} be a minimal IFS and $n > 1$ be an integer. Then the following are equivalent:*

- (1) *There is a continuous map $g : X \rightarrow \mathbb{Z}_n$ such that $(g \circ f_\lambda)(x) = g(x) + 1 \pmod{n}$, for all $\lambda \in \Lambda$.*
- (2) *There is a proper subset M in X which is minimal for \mathcal{F}^n and is not minimal for \mathcal{F}^t , for all $t < n$*

Proof. (1) \Rightarrow (2) Consider the map $g : X \rightarrow \mathbb{Z}_n$ such that $(g \circ f_\lambda)(x) = g(x) + 1 \pmod{n}$, for all $\lambda \in \Lambda$ and put $M_i = \{x \in X : g(x) = i\}$ for all $0 \leq i \leq n-1$. So M_0, M_1, \dots, M_{n-1} are pairwise disjoint nonempty sets. Take arbitrary elements $1 \leq i \leq n-1$ and $\lambda \in \Lambda$. Let $x \in M_i$. Since $(g \circ f_\lambda)(x) = g(x) + 1 \pmod{n}$ we have $f_\lambda(x) \in M_{i+1}$. Then $\mathcal{F}(M_i) \subset M_{i+1}$. On the other hand, let y be an arbitrary point in M_{i+1} . Since f_λ is a surjective map, there exists $x \in X$ such that $y = f_\lambda(x)$. So, $g(y) = g(f_\lambda(x)) = g(x) + 1 = i + 1$. Hence $x \in M_i$. This implies that $M_{i+1} \subseteq f_\lambda(M_i)$. So, $\mathcal{F}(M_{n-1}) = M_0$ and $\mathcal{F}(M_i) = M_{i+1}$, for all $0 \leq i < n-1$. This implies that M_0 is a \mathcal{F}^n -minimal set and is not minimal for \mathcal{F}^t , where $t < n$.

(2) \Rightarrow (1) Let M be a minimal set for \mathcal{F}^n such that M is not minimal for \mathcal{F}^t , for all $t < n$. So by statement (1) of Lemma 3.2, X is a disjoint union of $M, \mathcal{F}(M), \dots, \mathcal{F}^{n-1}(M)$. Now we define $g : X \rightarrow \mathbb{Z}_n$ by $g(x) = i$ if $x \in \mathcal{F}^i(M)$, where $\mathcal{F}^0(M) = M$. Because of the statement 2 in Lemma 3.2, g is continuous. \square

Definition 3.4. Let $\mathcal{A} = \{X_1, \dots, X_n\}$ be a finite partition of X . We say that the sets X_1, \dots, X_n are cyclically permuted by \mathcal{F} , if $f_\lambda(X_n) = X_1$ and $f_\lambda(X_i) = X_{i+1}$, for all $1 \leq i \leq n-1$ and all $\lambda \in \Lambda$.

It is clear that if the sets X_1, \dots, X_n are cyclically permuted by \mathcal{F} , then $\mathcal{F}_{\sigma_j}(X_i) = X_{i+j \pmod n}$ for all $\sigma \in \Lambda^{\mathbb{Z}_+}$ and all $i \geq 0$.

Next theorem is one of the main results of this paper. This theorem is an extension of Theorem 2.3 of [5] to iterated function systems and is proved the same way.

Theorem 3.5. Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers greater than 1. Let $m_i = j_1 \cdot j_2 \cdot \dots \cdot j_i$ for each i . Then, for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, \mathcal{F}_σ is topologically conjugate to g_α if and only if the following properties hold:

- (1) For each positive integer i , there is a cover P_i of X consisting of m_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by \mathcal{F} .
- (2) For each positive integer i , P_{i+1} partitions P_i .
- (3) If $V_1 \supset V_2 \supset V_3 \supset \dots$ is a nested sequence with $V_i \in P_i$ for each i , then $\bigcap_{i=1}^\infty V_i$ consists of a single point.

Proof. (\Rightarrow) Consider $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that \mathcal{F}_σ is topological conjugate with $g_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$, so there exist $h : X \rightarrow \Delta_\alpha$ such that $h \circ \mathcal{F}_{\sigma_n} = g_\alpha^n \circ h$ for all $n \geq 1$. Consider $Q_i = \{Y_{i,0}, \dots, Y_{i,m_i-1}\}$ and $P_i = \{X_{i,0}, \dots, X_{i,m_i-1}\}$, defined in the proof of Theorem 2.3. in [5]. So, for each $0 \leq j \leq m_{i-1}$,

$$\mathcal{F}_{\sigma_j}(X_{i,0}) = h^{-1} \circ g_\alpha^j \circ h(X_{i,0}) = h^{-1} \circ g_\alpha^j(Y_{i,0}) = h^{-1}(Y_{i,j}) = X_{i,j}.$$

The other properties are similar to proof of Theorem 2.3. [5]

(\Leftarrow) By the proof of Theorem 2.3 in [5], the set $\bigcap_{i \geq 1} Y_{i,t_i}$ is a singleton, where $t_i \in \{0, \dots, m_i - 1\}$, for all $i \geq 1$.

This property enable us to define $h : X \rightarrow \Delta_\alpha$ as follows.

Take $x \in X$. For each $i \geq 1$, there is a unique $k_i \leq m_i - 1$, such that

$x \in X_{i,k_i}$. On the other hand, $\cap_{i \geq 1} Y_{i,k_i}$ consists a single point y . Then, we can put $h(x) = y$. Again, by proof of Theorem 2.3. in [5], h is continuous and bijective. Now, we show that $h \circ \mathcal{F}_{\sigma_n} = g_\alpha^n \circ h$ for all $n \geq 1$.

Let $x \in X$. For each $i \geq 1$, there are unique $X_{i,k_i} \in P_i$ containing x and unique $t_i \leq m_i - 1$ congruent to $k_i + 1$ modulo m_i . Then, each of the points $h \circ \mathcal{F}_{\sigma_1}(x)$ and $g_\alpha \circ h(x)$ is an element of $\cap_{i \geq 1} Y_{i,t_i}$. Since the set $\cap_{i \geq 1} Y_{i,t_i}$ consists a single point, we have $h \circ \mathcal{F}_{\sigma_1}(x) = g_\alpha \circ h(x)$.

Repeating the above technique with $f_{\lambda_1}(x)$ instead of x , we get

$$h \circ \mathcal{F}_{\sigma_2}(x) = h(f_{\lambda_2}(f_{\lambda_1}(x))) = g_\alpha \circ h((f_{\lambda_1}(x))) = g_\alpha(g_\alpha(h(x))).$$

So, by induction on n we have

$$h(\mathcal{F}_{\sigma_n}(x)) = h(f_{\lambda_n}(\mathcal{F}_{\sigma_{n-1}}(x))) = g_\alpha(h(\mathcal{F}_{\sigma_{n-1}}(x))) = g_\alpha(g_\alpha^{n-1}(h(x))) = g_\alpha^n(h(x)).$$

□

The symmetric tent map $T_a : [0, 1] \rightarrow [0, 1]$ with $a \in [0, 2]$ is defined as follows:

$$T_a(x) = \begin{cases} ax & \text{if } 0 \leq x < \frac{1}{2}, \\ -ax + a & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Theorem 3.6. [1] *Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers greater than 1. The set of parameters a , such that for the tent map T_a the restriction of T_a to the closure of the orbit of $c = \frac{1}{2}$ is topologically conjugate to $g_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$, is dense in $[\sqrt{2}, 2]$.*

So, we have the following result.

Corollary 3.7. *Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers greater than 1 and $m_i = j_1 \cdot j_2 \cdot \dots \cdot j_i$ for each i . Let X be a compact Hausdorff space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an IFS with the following properties:*

(1) *For each positive integer i , there is a cover P_i of X consisting of m_i pairwise*

disjoint, nonempty, clopen sets which are cyclically permuted by \mathcal{F} .

(2) For each positive integer i , P_{i+1} partitions P_i .

(3) If $V_1 \supset V_2 \supset V_3 \supset \dots$ is a nested sequence with $V_i \in P_i$ for each i , then $\bigcap_{i=1}^{\infty} V_i$ consists of a single point.

Then the set of parameters s for which the restriction of T_s to the closure of the orbit of $c = \frac{1}{2}$ is topologically conjugate to \mathcal{F}_σ , for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, is dense in $[\sqrt{2}, 2]$.

Conditions (1) – (3) in Theorem 3.5 imply that X is a zero dimensional set.

So we have the following result:

Corollary 3.8. *Let X be a compact metric space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an **IFS**. If there exists an adding machine map g_α such that for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, \mathcal{F}_σ is topologically conjugate to g_α then X is a zero dimensional space.*

Let f be a continuous map on X and Q be an open cover of X . Let $N(Q)$ denote the minimal cardinality of a subcover of Q . The join of open covers Q_1, \dots, Q_n is

$$\bigvee_{k=1}^n = \{X_1, \dots, X_n; X_k \in Q_k, 1 \leq k \leq n\}$$

and put $f^{-1}(Q) = \{f^{-1}(V); V \in Q\}$. Define the topological entropy of f with respect to an open cover Q by $h(f|Q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\bigvee_{k=1}^n f^{-k}(Q))$ and the topological entropy of f by $h(f) = \sup_Q h(f|Q)$ where the supremum is taken over all open covers Q of X .

In [[13]Theorem 11.3.13], the authors prove that if f is a continuous map of an interval with zero topological entropy and S is a closed topological transitive invariant set without periodic orbits, then the restriction of f to S is topologically conjugate to an adding machine map.

Remark 3.9. Let \mathcal{F} be an IFS containing only one mapping f which satisfies the conditions (1)-(3) in Theorem 3.5 and consider the sequence $P_1 \preceq P_2 \preceq \dots$

of open covers in these statements. This is clear that $f^{-k}(P_i) = P_i$ for all $k, i \geq 1$, consequently $h(f|Q) = 0$, for all $i \geq 1$.

Now, let Q be a finite open cover of X . Statement (3) of Theorem 3.5 implies that there exists $i_0 \geq 1$ such that P_{i_0} is a refinement of Q . Then $h(f|Q) \leq h(f|P_{i_0}) = 0$ and consequently $h(f) = 0$. For more details see Theorem 11.3.13 of [13], Corollary 5.4.8 of [10] and Page 57 of [14].

4. MINIMAL IFS WITH REGULARLY RECURRENT POINTS AND ADDING MACHINE MAPS

Suppose that \mathcal{F} is minimal. Denote by $NM(\mathcal{F})$ the set of positive integers i such that some subset M of X is \mathcal{F}^i -minimal but is not \mathcal{F}^j -minimal for $j = 1, \dots, i-1$.

By Lemma 3.2, for every $n \in NM(\mathcal{F})$ there is a unique cover C_n of X which consists of the n -pairwise disjoint \mathcal{F}^n -minimal sets.

Definition 4.1. The point $x \in X$ is said to be regularly recurrent if for every neighborhood U of x , there is a positive integer n such that $\mathcal{F}_{\sigma_{ni}}(x) \in U$, for every $i \geq 0$ and every $\sigma \in \Lambda^{\mathbb{Z}_+}$.

A point $x \in X$ is called a periodic point of the IFS \mathcal{F} if there is a finite sequence $\lambda_1, \lambda_2, \dots, \lambda_n$ of elements in Λ such that $f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_1}(x) = x$ [2]. Our next lemmas are used in the proof of Theorem 4.6.

Lemma 4.2. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, if X is infinite and \mathcal{F} has some non-periodic regularly recurrent points then $S(\mathcal{F})$ is infinite.*

Proof. Fix $l \geq 1$. We show that there exists $n \in NM(\mathcal{F})$ such that $n > l$.

Let x be a non-periodic regularly recurrent point in X . There exists a neighborhood $U \subset X$ of x such that $\mathcal{F}_{\sigma_i}(x) \notin \overline{U}$, for each $\sigma \in \Lambda^{\mathbb{Z}_+}$ and $i = 1, \dots, l$.

Since x is a regularly recurrent point, there is a positive integer t such that for each non-negative integer i , and each $\sigma \in \Lambda^{\mathbb{Z}_+}$ we have $\mathcal{F}_{\sigma_{ti}}(x) \in U$.

So Lemma 3.2 implies that the set $M = \overline{\{\mathcal{F}_{\sigma_{ti}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i > 1}}$ is \mathcal{F}^t -minimal.

Consider n as the least positive integer for which M is \mathcal{F}^n -minimal. So, $n \in S$ and $M \subset \overline{U}$, hence $n > l$. \square

Lemma 4.3. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, X is infinite and \mathcal{F} has some non-periodic regularly recurrent points. Suppose that $l, n \in S$ and l is a multiple of n , then C_l refines C_n .*

Proof. Let $M \in C_l$ and $x \in M$. Since $\overline{\{\mathcal{F}_{\sigma_{li}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 0}} \subset M$ is a \mathcal{F}^l -minimal set. So $M = \overline{\{\mathcal{F}_{\sigma_{li}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 0}}$. This is clear that $\{\mathcal{F}_{\sigma_{li}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 0} \subset \{\mathcal{F}_{\sigma_{ni}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 0}$. So M is contained in $\overline{\{\mathcal{F}_{\sigma_{ni}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 0}}$ which is a \mathcal{F}^l -minimal set and consequently an element of C_n . \square

Lemma 4.4. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, X is infinite and \mathcal{F} has some non-periodic regularly recurrent points. Let $l \in S$ and n be a positive integer. If l is a multiple of n , then $n \in \text{NM}(\mathcal{F})$.*

Proof. Let $l = kn$ and M be a \mathcal{F}^l -minimal set. Put $M_1 = M \cup \mathcal{F}^n(M) \cup \dots \cup \mathcal{F}^{(k-1)n}(M)$. So, for all $1 \leq i \leq n-1$, $\mathcal{F}^i(M_1) \cap M_1 = \emptyset$ and these facts that $\mathcal{F}^l(M) \subseteq M$ and $l = kn$ imply that $\mathcal{F}^n(M_1) \subseteq \bigcup_{j=1}^k \mathcal{F}^{jn}(M) \subseteq M_1$, which completes the proof. \square

Lemma 4.5. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, X is infinite and \mathcal{F} has some non-periodic regularly recurrent points. Let l and n be two prime numbers in $\text{NM}(\mathcal{F})$. Then $ln \in \text{NM}(\mathcal{F})$.*

Proof. Let M_1 be a \mathcal{F}^l -minimal set and $x \in M_1$ be arbitrary. There exists a \mathcal{F}^n -minimal set M_2 such that, $x \in M_2$. Consider $A = M_1 \cap M_2$. If $\mathcal{F}^t(A) \cap A \neq \emptyset$, for some $t > 0$, then $\mathcal{F}^t(M_2) \cap M_2 \neq \emptyset$. This implies that t

is a multiple of n . By similar argument for M_1 we have $\mathcal{F}^{ln}(A) \cap A \neq \emptyset$ and $\mathcal{F}^i(A) \cap A = \emptyset$ for all $1 \leq i \leq ln - 1$. \square

In [[5] Theorem 2.4.], the authors prove that:

If $f : X \rightarrow X$ is minimal, X is infinite and f has some non-periodic regularly recurrent points, then there is a sequence α of prime numbers and a continuous surjective map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f$, for all $\lambda \in \Lambda$.

In the next theorem, we have a similar result for iterated function systems.

Theorem 4.6. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is minimal, X is infinite and \mathcal{F} has some non-periodic regularly recurrent points. Then there is a sequence α of prime numbers and a continuous surjective map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f_\lambda$, for all $\lambda \in \Lambda$. Also, $\pi^{-1}(\pi(x)) = \{x\}$, for every regularly recurrent point $x \in X$.*

Proof. Let p be a prime number and $s \in NM(\mathcal{F})$. Consider $N(s, p)$ as the multiplicity of p in the prime factorization of s and $N(p) = \max_{s \in S} N(s, p)$. Now assume that $\alpha = (p_1, p_2, \dots)$ is a sequence of primes such that the number of appearance of each prime number p appears in this sequence is exactly $N(p)$. Therefore if we put $n_i = p_1 p_2 \dots p_i$ then Lemmas 4.4 and 4.5 imply that $n_i \in NM(\mathcal{F})$, for every positive integers i . Consider $\{X_{i,1}, \dots, X_{i,n_i}\}$ as C_{n_i} cover of X . By Lemma 3.2, the statement (1) of Theorem 3.5 holds and by Lemma 3.2, the statement (2) of Theorem 3.5 holds. So by Theorem 3.5 for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, there exists a continuous surjective map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha^n \circ \pi = \pi \circ \mathcal{F}_{\sigma_n}$ for all $n \geq 1$. Now we show that for every regularly recurrent point $x \in X$, $\pi^{-1}(\pi(x)) = \{x\}$.

Let $y \in \pi^{-1}(\pi(x))$ and $y \neq x$. So there exists a sequence $V_1 \supset V_2 \supset V_3 \supset \dots$ with $V_i \in \{X_{i,1}, X_{i,2}, \dots, X_{i,n_i}\}$ for each i , such that $x, y \in \cap_{i \geq 1} V_i$.

Let V be an open subset of X such that $x \in V$ and $y \notin \overline{V}$. Since x is regularly recurrent there is a positive integer n such that $\mathcal{F}_{\sigma_{ni}}(x) \in V$ for all $i \geq 1$.

So $M = \{\mathcal{F}_{\sigma_{ni}}(x)\}_{\sigma \in \Lambda^{\mathbb{Z}_+}, i \geq 1}$ is an \mathcal{F}^n -minimal set in \overline{V} . This immediately implies that there exists $1 \leq t \leq n$ such that $t \in NM(\mathcal{F})$ and M is \mathcal{F}^t -minimal set. On the other hand, there is a positive integer j such that n_j is a multiple of t . So, by Lemma 4.3 and this fact that $x \in V_j \cap M$ we have $V_j \subset M$, which contradicts our assumption that $y \in V_j$ and $y \notin M$. Thus $\pi^{-1}(\pi(x)) = \{x\}$. \square

Corollary 4.7. *The IFS \mathcal{F} is minimal and each point of X is regularly recurrent if and only if there is a sequence α of prime numbers such that \mathcal{F}_σ is topologically conjugate to g_α , for every $\sigma \in \Lambda^{\mathbb{Z}_+}$.*

Let $g : X \rightarrow X$ be a continuous map, the ω -limit set of g at y is

$$\omega(y, g) = \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} g^k(y)}.$$

Definition 4.8. [6] Let X be a compact metric space, with $\alpha = (j_1, j_2, \dots)$ a sequence of integers so that $j_i \geq 2$ for each i . Denote by $S_\alpha(X)$ the set of all pairs (y, g) in $X \times C(X, X)$ such that $(\omega(y, g), g)$ is topologically conjugate to $g_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$.

Corollary 4.9. [5] *Let $\beta = (j_1, j_2, \dots)$ and $\gamma = (k_1, k_2, \dots)$ be sequences of integers with $j_i \geq 2$ and $k_i \geq 2$ for each i . We let M_β denote a function whose domain is the set of all prime numbers and which maps to the extended natural numbers $0, 1, 2, \dots, \infty$. The function M_β is defined by $M_\beta(p) = \sum_{i=1}^{\infty} n_i$ where n_i is the power of the prime p in the prime factorization of j_i . Then f_β and f_γ are topologically conjugate if and only if the functions M_β and M_γ are equal.*

Then we have the following corollary.

Corollary 4.10. *Suppose that the IFS \mathcal{F} is minimal and each point of X is regularly recurrent. There is a sequence α of prime numbers such that for*

every sequence β of prime numbers, $M_\beta = M_\alpha$ implies that \mathcal{F}_σ is topologically conjugate to g_β , for every $\sigma \in \Lambda^{\mathbb{Z}_+}$.

Let $\alpha = (j_1, j_2, \dots)$ a sequence of integers so that $j_i \geq 2$ for each i . In [7], the authors prove that if for every prime number p , $M_\alpha(p) < \infty$ then $S_\alpha(X)$ is a nowhere dense subset of $X \times C(X, X)$.

So, by considering Lemmas 4.3 and 4.4 and the sequence α in the proof of Theorem 4.6 we have the following result.

Corollary 4.11. *Suppose that the IFS \mathcal{F} is minimal and each point of X is regularly recurrent. There exists a sequence α of prime numbers such that the set of all $(y, g) \in X \times C(X, X)$ which $g : \omega(y, g) \longrightarrow \omega(y, g)$, for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, is topologically conjugate to \mathcal{F}_σ is a nowhere dense subset of $M \times C(M, M)$.*

In [[15], Corollary 4.3] the authors prove that if $f : X \rightarrow X$ is a sensitive non-wandering dynamical system with the shadowing property then the set of non-periodic regularly points of f is dense in X . So we have the following result.

Theorem 4.12. *Suppose that $f : X \rightarrow X$ is sensitive and minimal map with the shadowing property. Then there is a sequence α of prime numbers and a continuous map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f$ and π is one to one on a dense subset of X .*

Proof. Minimality of f implies that $\overline{\{f^n(x)\}_{n \geq 0}} = X$ for all $x \in X$. So f is a non-wandering system and does not have any periodic point. Then by Corollary 4.3. of [15] the set of all regularly recurrent points is dense in X . Consequently Theorem 4.6 (in the case of Λ has only one element), implies that there is a sequence α of prime numbers and a continuous map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f$ and π is one to one on the set of all regularly recurrent points which a dense subset of X . \square

So it seems plausible to put forward the following conjectures.

Conjecture 1: Suppose that $f : X \rightarrow X$ is a sensitive minimal map with the shadowing property. Then there is a sequence α of prime numbers and a homeomorphism map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f$.

Conjecture 2: Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is a sensitive minimal IFS map with the shadowing property. Then there is a sequence α of prime numbers and a homeomorphism map $\pi : X \rightarrow \Delta_\alpha$ such that $g_\alpha \circ \pi = \pi \circ f_\lambda$, for all $\lambda \in \Lambda$.

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